

Note on the Stieltjes constants: series with Stirling numbers of the first kind

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Abstract

The Stieltjes constants $\gamma_k(a)$ appear as the coefficients in the regular part of the Laurent expansion of the Hurwitz zeta function $\zeta(s, a)$ about $s = 1$. We generalize the integral and Stirling number series results of [4] for $\gamma_k(a = 1)$. Along the way, we point out another recent asymptotic development for $\gamma_k(a)$ which provides convenient and accurate results for even modest values of k .

Key words and phrases

Stieltjes constants, Riemann zeta function, Hurwitz zeta function, Laurent expansion, digamma function, polygamma function, harmonic numbers

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Introduction and statement of results

The Stieltjes (or generalized Euler) constants $\gamma_k(a)$ appear as expansion coefficients in the Laurent series for the Hurwitz zeta function $\zeta(s, a)$ about its simple pole at $s = 1$ [5, 6, 10, 16, 20, 22, 24],

$$\zeta(s, a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(a) (s-1)^n. \quad (1.1)$$

These constants are important in analytic number theory and elsewhere, where they appear in various estimations and as a result of asymptotic analyses, being given by the limit relation

$$\gamma_k(a) = \lim_{N \rightarrow \infty} \left[\sum_{j=1}^N \frac{\ln^k(j+a)}{j} - \frac{\ln^{k+1}(N+a)}{k+1} \right].$$

In particular, $\gamma_0(a) = -\psi(a)$, where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function, with $\Gamma(z)$ the Gamma function. With γ the Euler constant and $\gamma_1 = \gamma_1(1)$ and $\gamma_2 = \gamma_2(1)$, we recall the connection with sums of reciprocal powers of the nontrivial zeros ρ of the Riemann zeta function,

$$\sum_{\rho} \frac{1}{\rho^2} = 1 - \frac{\pi^2}{8} + 2\gamma_1 + \gamma^2, \quad \sum_{\rho} \frac{1}{\rho^3} = 1 - \frac{7}{8}\zeta(3) + \gamma^3 + 3\gamma\gamma_1 + \frac{3}{2}\gamma_2,$$

such relations following from the Hadamard factorization. We recall the connection of differences of Stieltjes constants with logarithmic sums,

$$\gamma_{\ell}(a) - \gamma_{\ell}(b) = \sum_{n=0}^{\infty} \left[\frac{\ln^{\ell}(n+a)}{n+a} - \frac{\ln^{\ell}(n+b)}{n+b} \right]. \quad (1.2)$$

An effective asymptotic expression for γ_k [17] and $\gamma_k(a)$ [18] for $k \gg 1$ has recently been given. From these expressions, which show accurate estimations for even modest

values of k , previously known results on sign changes within the sequence of Stieltjes constants follow. The asymptotic expressions encapsulate both the magnitude $|\gamma_m(a)|$ and the changes in sign of the sequence $\{\gamma_m(a)\}$. Additionally in [18], an elaborate analysis is provided for a certain alternating binomial sum of the Stieltjes constants.

Evaluations of the first and second Stieltjes constants at rational argument have been given very recently [3, 12]. These decompositions are effectively Fourier series, thus implying many extensions and applications, and they supplement the relations presented in [7]. Besides elaborating on a multiplication formula for the Stieltjes constants, [12] provides examples of integrals evaluating in terms of differences of the first and second of these constants. In addition, presented there is a novel method of determining log-log integrals with a certain polynomial denominator integrand.

In this note, we briefly describe how the integral and series representations in [4] for $\gamma_k(a=1)$ may be readily generalized to $\gamma_k(a)$. Having the parameter a is very useful, as even from the $\gamma_0(a)$ case representations then follow for $\ln \Gamma(a)$, the polygamma functions, and, using integer arguments, harmonic and generalized harmonic numbers. The key starting point integral representation of these developments is given, so that proofs are largely omitted.

The Hurwitz zeta function, initially defined by $\zeta(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s}$ for $\text{Re } s > 1$, has an analytic continuation to the whole complex plane [2, 23]. In the case of $a = 1$, $\zeta(s, a)$ reduces to the Riemann zeta function $\zeta(s)$ [13, 21]. In this instance, by convention, the Stieltjes constants $\gamma_k(1)$ are simply denoted γ_k [5, 16, 19, 20, 25]. We

recall that $\gamma_k(a+1) = \gamma_k(a) - (\ln^k a)/a$, and more generally that for $n \geq 1$ an integer

$$\gamma_k(a+n) = \gamma_k(a) - \sum_{j=0}^{n-1} \frac{\ln^k(a+j)}{a+j},$$

as follows from the functional equation $\zeta(s, a+n) = \zeta(s, a) - \sum_{j=0}^{n-1} (a+j)^{-s}$. In fact, an interval of length $1/2$ is sufficient to characterize the $\gamma_k(a)$'s [15].

Unless specified otherwise below, letters j, k, ℓ, m, n , and r denote positive integers. The Euler constant is given by $\gamma = -\psi(1) = \gamma_0(1)$. The polygamma functions are denoted $\psi^{(n)}(z)$ and we note that $\psi^{(n)}(z) = (-1)^{n+1} n! \zeta(n+1, z)$ [1, 14]. We let $s(n, k)$ denote the Stirling numbers of the first kind. We note the cutoff property $s(n, k) = 0$ for $n < k$, consistent with the combinatorial interpretation of $(-1)^{n+k} s(n, k)$ as the number of permutations of n symbols having k cycles.

Reference [4] does not seem to be aware of the representation [8]

$$\gamma_k(a) = \frac{1}{2a} \ln^k a - \frac{\ln^{k-1} a}{k+1} + \frac{1}{a} \int_0^\infty \frac{(y/a - i) \ln^k(a - iy) + (y/a + i) \ln^k(a + iy)}{(1 + y^2/a^2)(e^{2\pi y} - 1)} dy. \quad (1.3)$$

From this integral representation the following results follow.

Proposition 1. Let $\text{Re } a > 0$. Then (a)

$$\gamma_0(a) = -\psi(a) = \frac{1}{2a} - \ln a + \frac{1}{2\pi i a} \int_0^1 \left[\frac{1}{1 - \frac{\ln(1-u)}{2\pi i a}} - \frac{1}{1 + \frac{\ln(1-u)}{2\pi i a}} \right] \frac{du}{u},$$

(b)

$$\gamma_0(a) = \frac{1}{2a} - \ln a - \frac{1}{\pi a} \sum_{k=0}^{\infty} \frac{(2k+1)!(-1)^k}{(2\pi a)^{2k+1}} \sum_{n=1}^{\infty} \frac{s(n, 2k+1)(-1)^n}{n!n},$$

and (c)

$$\gamma_0(a) = \frac{1}{2a} - \ln a - \frac{1}{2\pi^2 a^2} \sum_{n=0}^{\infty} \frac{(2n+1)!(-1)^n}{(2\pi a)^{2n}} \zeta(2n+2).$$

Proposition 2. Let $\operatorname{Re} a > 0$. Then (a)

$$\gamma_1(a) = \frac{\ln a}{2a} - \frac{1}{2} \ln^2 a + \ln a \left[-\psi(a) - \frac{1}{2a} + \ln a \right]$$

$$- \frac{1}{\pi a} \sum_{n=1}^{\infty} \frac{1}{nn!} \sum_{k=0}^{[n/2]} \frac{(-1)^k}{(2\pi a)^{2k+1}} |s(2k+2, 2)| |s(n, 2k+1)|,$$

wherein $s(2k+2, 2) = (2k+1)!H_{2k+1}$, $H_n = \sum_{k=1}^n 1/k$ being the n th harmonic number, and (b)

$$\gamma_m(a) = \frac{\ln^m a}{2a} - \frac{1}{m+1} \ln^{m+1} a$$

$$+ \frac{1}{2\pi i} \int_0^1 \left[\frac{\ln^m \left(a - \frac{\ln(1-u)}{2\pi i} \right)}{a - \frac{\ln(1-u)}{2\pi i}} - \frac{\ln^m \left(a + \frac{\ln(1-u)}{2\pi i} \right)}{a + \frac{\ln(1-u)}{2\pi i}} \right] \frac{du}{u}.$$

As we discuss, Proposition 1 may be considered the expansion of $\gamma_0(a)$ or of the digamma function about $|a| \rightarrow \infty$ with $\arg a < \pi$.

Discussion

The proofs of Propositions 1 and 2 follow from (1.3) and the approach of [4]. Rather than provide such details, we give a complementary illustrating verification of Proposition 1. For this purpose we recall the generating function (e.g., [1])

$$\sum_{n=0}^{\infty} s(n, k) \frac{x^n}{n!} = \frac{1}{k!} \ln^k(1+x), \quad (2.1)$$

holding for $|x| < 1$ and $|x| = 1$ but excluding $x = -1$.

Proof of Proposition 1. By geometric series, the right side of (a) is given by

$$\frac{1}{2a} - \ln a + \frac{1}{2\pi i a} \int_0^1 \left[\frac{1}{1 - \frac{\ln(1-u)}{2\pi i a}} - \frac{1}{1 + \frac{\ln(1-u)}{2\pi i a}} \right] \frac{du}{u}$$

$$= \frac{1}{2a} - \ln a + \frac{2}{2\pi ia} \int_0^1 \sum_{\substack{j=0 \\ j \text{ odd}}}^{\infty} \frac{\ln^j(1-u)}{(2\pi ia)^j} \frac{du}{u},$$

the integral being evaluated with a change of variable,

$$\int_0^1 \frac{\ln^j(1-u)}{u} du = (-1)^{j+1} \int_0^{\infty} \frac{v^j}{e^v - 1} dv = (-1)^{j+1} j! \zeta(j+1).$$

Now [14] (p. 943)

$$\psi(z) = \ln z - \frac{1}{2a} - 2 \int_0^{\infty} \frac{t dt}{(t^2 + z^2)(e^{2\pi t} - 1)}, \quad \operatorname{Re} z > 0.$$

Then again by the use of geometric series,

$$\begin{aligned} \psi(z) &= \ln z - \frac{1}{2a} - \frac{1}{2\pi^2 z^2} \sum_{j=0}^{\infty} (-1)^j \int_0^{\infty} \frac{v^{2j+1}}{(4\pi^2 z^2)^j (e^v - 1)} dv \\ &= \ln z - \frac{1}{2a} - \frac{1}{2\pi^2 z^2} \sum_{j=0}^{\infty} \frac{(-1)^j}{(4\pi^2 z^2)^j} (2j+1)! \zeta(2j+2). \end{aligned}$$

Thus parts (a) and (c) follow.

Part (b) follows from geometric series expansion of the right side of (a) and the use of the generating function (2.1). In particular, the term with $s(0, j) = \delta_{0j}$, the Kronecker symbol, does not contribute. \square

Remarks. By reordering the double sum and using the cutoff property of Stirling numbers of the first kind, part (b) may be rewritten as

$$\gamma_0(a) = \frac{1}{2a} - \ln a - \frac{1}{\pi a} \sum_{n=1}^{\infty} \frac{1}{n! n} \sum_{k=0}^{[n/2]} \frac{(2k+1)! (-1)^k}{(2\pi a)^{2k+1}} s(n, 2k+1) (-1)^n.$$

The Stirling numbers of the first kind may be written with the generalized harmonic numbers, and the first few are given by $s(n+1, 1) = (-1)^n n!$, $s(n+1, 2) =$

$(-1)^{n+1}n!H_n$, $s(n+1, 3) = (-1)^n \frac{n!}{2}[H_n^2 - H_n^{(2)}]$, and $s(n+1, 4) = (-1)^{n+1} \frac{n!}{6}[H_n^3 - 3H_n H_n^{(2)} + 2H_n^{(3)}]$. The generalized harmonic numbers in terms of polygamma function values

$$H_n^{(r)} = \sum_{k=1}^n \frac{1}{k^r} = \frac{(-1)^{r-1}}{(r-1)!} [\psi^{(r-1)}(n+1) - \psi^{(r-1)}(1)] = \frac{(-1)^{r-1}}{(r-1)!} \int_0^1 \frac{(t^n - 1)}{t - 1} \ln^{r-1} t \, dt$$

enter the representations of Proposition 2 for the higher Stieltjes constants.

References

- [1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Washington, National Bureau of Standards (1964).
- [2] B. C. Berndt, On the Hurwitz zeta function, Rocky Mtn. J. Math. **2**, 151-157 (1972).
- [3] I. V. Blagouchine, A theorem for the closed-form evaluation of the first generalized Stieltjes constant at rational arguments, arXiv:1401.3724v2 (2014).
- [4] I. V. Blagouchine, Expansions of the generalized Euler's constants into the series of polynomials in π^{-2} and into the formal enveloping series with rational coefficients only, arXiv:1501.00740v2.
- [5] W. E. Briggs, Some constants associated with the Riemann zeta-function, Mich. Math. J. **3**, 117-121 (1955).
- [6] M. W. Coffey, New results on the Stieltjes constants: Asymptotic and exact evaluation, J. Math. Anal. Appl. **317**, 603-612 (2006); arXiv:math-ph/0506061.
- [7] M. W. Coffey, On representations and differences of Stieltjes coefficients, and other relations, Rocky Mtn. J. Math. **41**, 1815-1846 (2011), arXiv/math-ph/0809.3277v2 (2008).

- [8] M. W. Coffey, The Stieltjes constants, their relation to the η_j coefficients, and representation of the Hurwitz zeta function, *Analysis* **30**, 383 (2010), arXiv/math-ph/:0706.0343v2 (2007).
- [9] M. W. Coffey Series representations for the Stieltjes constants, *Rocky Mtn. J. Math.* **44**, 443-477 (2014), arXiv/math-ph/0905.1111 (2009).
- [10] M. W. Coffey, Certain logarithmic integrals, including solution of Monthly problem 11629, zeta values, and expressions for the Stieltjes constants, arXiv:1201.3393 (2012).
- [11] M. W. Coffey, Hypergeometric summation representations of the Stieltjes constants, *Analysis* **33**, 121-142 (2013), arXiv:1106.5148 (2011).
- [12] M. W. Coffey, Functional equations for the Stieltjes constants, to appear in *Ramanujan J.* (2015), arXiv:1402.3746 (2014).
- [13] H. M. Edwards, *Riemann's Zeta Function*, Academic Press, New York (1974).
- [14] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York (1980).
- [15] E. R. Hansen and M. L. Patrick, Some relations and values for the generalized Riemann zeta function, *Math. Comp.* **16**, 265-274 (1962).
- [16] J. C. Kluyver, On certain series of Mr. Hardy, *Quart. J. Pure Appl. Math.* **50**, 185-192 (1927).

- [17] C. Knessl and M. W. Coffey, An effective asymptotic formula for the Stieltjes constants, *Math. Comp.* **80**, 379-386 (2011).
- [18] C. Knessl and M. W. Coffey, An asymptotic form for the Stieltjes constants $\gamma_k(a)$ and for a sum $S_\gamma(n)$ appearing under the Li criterion, *Math. Comp.* **80**, 2197-2217 (2011).
- [19] R. Kreminski, Newton-Cotes integration for approximating Stieltjes (generalized Euler) constants, *Math. Comp.* **72**, 1379-1397 (2003).
- [20] D. Mitrović, The signs of some constants associated with the Riemann zeta function, *Mich. Math. J.* **9**, 395-397 (1962).
- [21] B. Riemann, Über die Anzahl der Primzahlen unter einer gegebenen Grösse, *Monats. Preuss. Akad. Wiss.*, 671 (1859-1860).
- [22] T. J. Stieltjes, Correspondance d'Hermite et de Stieltjes, Volumes 1 and 2, Gauthier-Villars, Paris (1905).
- [23] E. C. Titchmarsh, The Theory of the Riemann Zeta-Function, 2nd ed., Oxford University Press, Oxford (1986).
- [24] J. R. Wilton, A note on the coefficients in the expansion of $\zeta(s, x)$ in powers of $s - 1$, *Quart. J. Pure Appl. Math.* **50**, 329-332 (1927).
- [25] N.-Y. Zhang and K. S. Williams, Some results on the generalized Stieltjes constants, *Analysis* **14**, 147-162 (1994).